TIGHT BOUNDS FOR LINKAGES IN PLANAR GRAPHS

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joint work with Stavros Kolliopoulos, Philipp Krause, Daniel Lokshtanov, Saket Saurabh, Dimitrios Thilikos
Outline

1. Introduction: Disjoint paths and irrelevant vertices
2. Our results
3. Upper bound in planar graphs
4. Lower bound
5. Conclusion
Graphs are . . .

finite, undirected and simple.
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The size of an \((n \times n)\)-grid is \(n\).
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Subdivided \((7 \times 7)\)-grid

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The Disjoint Paths Problem

**DISJOINTPATHS**

**Input:** Graph $G$, terminals $(s_1, t_1), \ldots, (s_k, t_k) \in V(G)^{2k}$

**Question:** Are there $k$ pairwise vertex disjoint paths $P_1, \ldots, P_k$ in $G$ s.t. $P_i$ has endpoints $s_i$ and $t_i$?

Applications: chip design, routing, transportation and telecommunication networks, . . .

**Example**

*An instance of DISJOINTPATHS with $k = 2$:*

![Graph diagram]
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![Graph with terminals $S_1, T_1, S_2, T_2$ and disjoint paths between them.](image)
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… it’s a ‘no’-instance.
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*Example*

*An instance of DisjointPaths with $k = 2$:*

![Graph Diagram]

. . . it’s a ‘no’-instance.
Notice: there are two disjoint paths from the set \( \{ s_1, s_2 \} \) to \( \{ t_1, t_2 \} \).
Disjoint Paths Problem: Example

An instance of \textsc{DisjointPaths} with $k = 5$:
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An instance of DISJOINTPATHS with \( k = 5 \):

\[ \text{...it's a 'yes'-instance.} \]
**Disjoint Paths Problem**

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If $k$ is part of the input. . .

- **DISJOINTPATHS** is NP complete (Karp 1975)
- **DISJOINTPATHS** remains NP complete on planar graphs (Lynch 1975)
**Parameterized Disjoint Paths Problem**

<table>
<thead>
<tr>
<th>p-DISJOINTPATHS</th>
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<tbody>
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**Theorem (Robertson, Seymour 1983–today)**

$p$-DISJOINTPATHS has a cubic FPT algorithm: there is a computable function $f$ such that the algorithm runs in time $f(k) \cdot |V(G)|^3$. 
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But…

$f(k)$ is a huge tower of exponentiations$^1$! 😞

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**Source of huge $f(k)$**

- R&S: Graph Minors I (1983) – Graph Minors XXIII (2010).
- In Graph Minors XXII. Irrelevant vertices in linkage problems:

  *If $G$ contains a subdivided grid of size $g(k)$ for some huge function $g$, then the middle vertex $v$ of the grid is irrelevant to DPP.*

- $f(k) \geq g(k)$

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Let’s consider planar graphs from now on.

**Theorem (Reed, Robertson, Schrijver, Seymour 1991)**

On planar graphs: \texttt{p-DISJOINTPATHS} has a linear FPT algorithm: 
\[ f(k) \cdot |V(G)|. \]

\[ \ldots \text{where } f \text{ is a huge computable function. } \]

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\(^2\)László Lovász, Graph Minor Theory, 2005.
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**Definition**

For an instance $(G, (s_1, t_1), \ldots, (s_k, t_k))$, we say that a vertex $v \in V(G) \setminus \{ \text{terminals} \}$ is irrelevant, if

$$(G, (s_1, t_1), \ldots, (s_k, t_k)) \text{ is a ‘yes’-instance } \iff (G - v, (s_1, t_1), \ldots, (s_k, t_k)) \text{ is a ‘yes’-instance.}$$
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Given: planar G with terminals \((s_1, t_1), \ldots, (s_k, t_k)\).

What is the minimum size \(g(k)\) of a grid in G that guarantees us a vertex \(v\) that is irrelevant for \textsc{DisjointPaths}?
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Upper and lower bound

Theorem (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos)

Planar G with k pairs of terminals.
∃ function $g \in O\left(2^k\right)$ such that if G contains a subdivided $g(k) \times g(k)$ grid, then G contains an irrelevant vertex.

Theorem (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos)

$g(k) > 2^{k-1} + 1$:
There exists a planar graph G with k pairs of terminals, such that DISJOINT PATHS has a unique solution, and the solution uses all vertices of a $(2^{k-1} + 1) \times (2^{k-1} + 1)$-grid.
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Theorem (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos)

Planar $G$ with $k$ pairs of terminals.

There exists a function $g \in 2^{O(k)}$ such that if $G$ contains a subdivided $g(k) \times g(k)$ grid, then $G$ contains an irrelevant vertex.

$$g_{\text{exact}}(k) = 2^k \cdot 16\sqrt{2}k^{3/2}$$

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Planar DISJOINT PATHS

**Theorem (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos)**

$G$ planar with $k$ pairs of terminals.

There is an $O(|V(G)|^2)$ time algorithm that outputs an induced subgraph $G'$ of $G$ s.t.

- $G'$ is a ‘yes’-instance of DPP iff $G'$ is a ‘yes’-instance of DPP
- $\text{treewidth}(G') \leq 2^{c \cdot k}$

**Proof sketch.**

For every $v \in V(G)$

- If $v$ in center of a subdivided grid with $> 2^{c \cdot k}$ layers
  
  $G := G - v$

Return $G' := G$

**Corollary (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos)**

*Planar DISJOINT PATHS* can be solved in time $2^{2^{O(k)}} \cdot |V(G)|^2$. 
**Planar Disjoint Paths**

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Theorem (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos)

Planar G with k pairs of terminals.

\[ \exists \text{ function } g \in 2^{O(k)} \text{ such that if } G \text{ contains a subdivided } g(k) \times g(k) \text{ grid, then } G \text{ contains an irrelevant vertex.} \]

Proof Sketch. Assume G is a ‘yes’-instance containing a grid of size \( > 2^{c \cdot k} \). G embedded in the plane.

- Assume that no terminals are inside the grid
- Let \( \mathcal{P} \) be a solution that crosses as few layers of the grid as possible.
- Delete the subgraph of G within the perimeter of the grid, but put the paths in \( \mathcal{P} \) back
Upper bound: Proof Sketch

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Proof Sketch: grid size for irrelevant vertices

- the segments of paths of $\mathcal{P}$ within the perimeter form an outer-planar graph
- show: (a) the number of segment types (colors) is $\leq 4k - 3$,
  (b) there are at most $2^k$ segments of each type
- (a)+(b): bound on number of segments of $\mathcal{P}$ in the grid
- put the grid back and find $\mathcal{P}$ uses the outer $2^{c\cdot k}$ layers only – avoiding the middle vertex!
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Large grids guarantee irrelevant vertices

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Given a planar graph $G$ and terminals $(s_1, t_1), \ldots, (s_k, t_k) \in V(G)^{2k}$, what is the minimum size $g(k)$ of a grid in $G$ that guarantees us a vertex irrelevant for DISJOINTPATHS?

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There exists a planar graph $G$ with $k$ pairs of terminals, such that **DISJOINTPATHS** has a unique solution, and the solution uses all vertices of a $(2^{k-1} + 1) \times (2^{k-1} + 1)$-grid.
Large grids guarantee irrelevant vertices

**Question**

Given a planar graph $G$ and terminals $(s_1, t_1), \ldots, (s_k, t_k) \in V(G)^{2k}$, what is the minimum size $g(k)$ of a grid in $G$ that guarantees us a vertex irrelevant for DISJOINTPATHS?

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The lower bound: Proof

Instance of DISJOINTPATHS with \( k = 5 \):
The lower bound: Proof

Instance of \textsc{DisjointPaths} with $k = 5$: 
The lower bound: Proof

Instance of DISJOINTPATHS with $k = 5$: 

![Diagram showing an instance of DISJOINTPATHS with 5 paths. The diagram includes multiple paths connecting the source and target nodes, each path marked with a different color.](image-url)
Constructing the Example
Constructing the Example
Constructing the Example
Constructing the Example
Proof
Proof
Proof
Proof
Putting things together

Question
Given: planar $G$, terminals $(s_1, t_1), \ldots, (s_k, t_k)$.
What is the minimum size $g(k)$ of a grid in $G$ that guarantees us a vertex that is irrelevant for DISJOINTPATHS?

Theorem (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos, 2011)
$2^{k-1} + 1 < g(k) \leq 2^k \cdot 16\sqrt{2}k^{3/2}$.

Corollary
On planar graphs, DISJOINTPATHS can be solved in $2^{O(k)} \cdot |V(G)|^2$.

Best parameter dependency so far!
**Question**

Given: planar G, terminals $(s_1, t_1), \ldots, (s_k, t_k)$. What is the minimum size $g(k)$ of a grid in G that guarantees us a vertex that is irrelevant for \textsc{DisjoinPaths}?

**Theorem** (A., Krause, Kolliopoulos, Lokshtanov, Saurabh, Thilikos, 2011)

$2^{k-1} + 1 < g(k) \leq 2^k \cdot 16\sqrt{2k^{3/2}}$.

**Corollary**

On planar graphs, \textsc{DisjoinPaths} can be solved in $2^{2^{O(k)}} \cdot |V(G)|^2$.

Best parameter dependency so far!
**Putting things together**

**Question**
Given: planar $G$, terminals $(s_1, t_1), \ldots, (s_k, t_k)$.
What is the minimum size $g(k)$ of a grid in $G$ that guarantees us a vertex that is irrelevant for $\text{DISJOINTPATHS}$?

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On planar graphs, $\text{DISJOINTPATHS}$ can be solved in $2^{2^{O(k)}} \cdot |V(G)|^2$.

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Outline

1. Introduction: Disjoint Paths
2. R&S Algorithm – Irrelevant vertices are expensive
3. Price? Upper bound in planar graphs
4. Price? A lower bound
5. Conclusion
Future research:

- what about non-planar graphs?
- is there a faster algorithm for DISJOINTPATHS on planar graphs?
- improve parameter dependency of related algorithms (topological minor test, test for immersions)
Vielen Dank!